

Utility Functions

Part III - The Exponential Utility Function

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In this white paper we will define the exponential utility function. To that end we will work through the following hypothetical problem from Part I...

Our Hypothetical Problem

We are given the following investment payoffs and market assumptions...

Table 1: Investment Payoffs

Symbol	Payoff	Probability
W_1	200	0.40
W_2	500	0.30
W_3	700	0.20
W_4	950	0.10

Table 2: Market Assumptions

Symbol	Description	Value
μ	Annual risk-free interest rate (%)	4.50
κ	Annual expected investment return (%)	8.00
T	Investment term in years (#)	3.00

Questions:

1. Given a discount rate of 8.00%, what is the value of the certainty equivalent at time T and at time zero?
2. What is the value of the scalar alpha?
3. Graph utility, marginal utility, and risk aversion.

Expected Wealth

We will define the variable p_i to be the probability of realizing wealth W_i (the i'th random investment payoff). The equation for expected wealth is...

$$\mathbb{E}[W] = \sum_{i=1}^n p_i W_i \quad (1)$$

Using Equation (1) above and the data in Table 1 above, the expected increase in wealth from our investment as defined in Table 1 above is...

$$\mathbb{E}[W] = 200 \times 0.40 + 500 \times 0.30 + 700 \times 0.20 + 950 \times 0.10 = 465.00 \quad (2)$$

Exponential Utility Function

We will define the variable α to be a scalar whose value is greater than zero. The equation for our exponential utility function and it's first and second derivatives with respect to the random increase in wealth (W_i) are the following equations...

$$U(W_i) = 1 - \text{Exp}\left\{-\alpha W_i\right\} \text{ ...where... } U'(W_i) = \alpha \text{Exp}\left\{-\alpha W_i\right\} \text{ ...and... } U''(W_i) = -\alpha^2 \text{Exp}\left\{-\alpha W_i\right\} \quad (3)$$

Using Equation (3) above, we can make the following statements...

$$U(W_i) = 0 \text{ ...when... } W_i = 0 \text{ ...and... } U(W_i) > 0 \text{ ...when... } W_i > 0 \quad (4)$$

The equation for our initial guess of the value of scalar parameter α where W_{max} is the maximum investment payoff is...

$$\text{if... } 1 - \text{Exp}\left\{-\alpha W_{max}\right\} = 0.80 \text{ ...then... } \alpha = -\ln\left(1 - 0.80\right) / W_{max} \quad (5)$$

We will define the variable λ to be the Arrow-Pratt measure of risk aversion. Using Equation (3) above, the equation for this measure of risk aversion is... [1]

$$\lambda = -\frac{U''(W_i)}{U'(W_i)} = -\frac{-\alpha^2 \text{Exp}\{-\alpha W_i\}}{\alpha \text{Exp}\{-\alpha W_i\}} = \alpha \quad (6)$$

Note that the value of λ in Equation (6) above implies constant absolute risk aversion (CARA). Investors will invest the same percentage of their wealth in risky assets as they get wealthier.

Expected Utility

Using Equation (3) above, the equation for the expected utility of wealth is... [1]

$$\mathbb{E}\left[U(W)\right] = \sum_{i=1}^n p_i U(W_i) = \sum_{i=1}^n p_i \left(1 - \text{Exp}\left\{-\alpha W_i\right\}\right) = 1 - \sum_{i=1}^n p_i \text{Exp}\left\{-\alpha W_i\right\} \quad (7)$$

Using Equation (7) above, the equation for the derivative of the expected utility of wealth with respect to the scalar parameter α is...

$$\frac{\delta}{\delta \alpha} \mathbb{E}\left[U(W)\right] = \sum_{i=1}^n p_i W_i \text{Exp}\left\{-\alpha W_i\right\} \quad (8)$$

Certainty Equivalent

We will define the variable CE to be the value of the certainty equivalent at time T . The dollar value of the certainty equivalent is such that the utility of the certainty equivalent is equal to the utility of expected wealth. This statement in equation form is... [1]

$$U(CE) = \mathbb{E}\left[U(W)\right] \quad (9)$$

Using Equations (3) and (7) above, we can rewrite Equation (9) above as...

$$1 - \text{Exp}\left\{-\alpha CE\right\} = 1 - \sum_{i=1}^n p_i \text{Exp}\left\{-\alpha W_i\right\} \quad (10)$$

In Table 2 above we defined the variable μ to be the risk-free rate and the variable κ to be the risk-adjusted discount rate. If we are given the discount rate κ then using Equation (1) above the equation for the value of the certainty equivalent at time T is...

$$\text{if... } \kappa = \left(\mathbb{E}\left[W\right] / CE (1 + \mu)^{-T}\right)^{1/T} - 1 \text{ ...then... } CE = \mathbb{E}\left[W\right] \left(\frac{1 + \mu}{1 + \kappa}\right)^T \quad (11)$$

Given the value of the certainty equivalent at time T (Equation (11) above) we want to solve Equation (10) above for the value of the scalar α . We will make the following scalar parameter definitions...

$$\alpha = \text{Actual value of the scalar alpha} \text{ ...and... } \hat{\alpha} = \text{Guess value of the scalar alpha} \quad (12)$$

Using Equations (10) and (12) above, we will define the function $f(\alpha)$ to be...

$$f(\alpha) = \sum_{i=1}^n p_i \text{Exp}\left\{-\alpha W_i\right\} - \text{Exp}\left\{-\alpha CE\right\} = 0 \quad (13)$$

Using Equations (10) and (12) above, we will define the function $f(\hat{\alpha})$ to be...

$$f(\hat{\alpha}) = \sum_{i=1}^n p_i \text{Exp}\left\{-\hat{\alpha} W_i\right\} - \text{Exp}\left\{-\hat{\alpha} CE\right\} \quad (14)$$

Using Equation (8) above, the derivative of Equation (14) above with respect to the scalar α is...

$$f'(\hat{\alpha}) = CE \text{Exp}\left\{-\hat{\alpha} CE\right\} - \sum_{i=1}^n p_i W_i \text{Exp}\left\{-\hat{\alpha} W_i\right\} \quad (15)$$

To solve for κ , the Newton-Raphson equation that we will iterate is... [2]

$$\alpha + \hat{\epsilon} = \hat{\alpha} + \frac{f(\alpha) - f(\hat{\alpha})}{f'(\hat{\alpha})} \quad \left| \quad f(\alpha) = \text{Equation (13)}, f(\hat{\alpha}) = \text{Equation (14)}, f'(\hat{\alpha}) = \text{Equation (15)} \right. \quad (16)$$

Answers To Our Hypothetical Problem

1. Given a discount rate of 8.00%, what is the value of the certainty equivalent at time T and at time zero?

Using Equations (2) and (11) above and the data in Table 1 above, the value of the certainty equivalent at time T is...

$$CE = 465.00 \times \left(\frac{1 + 0.0450}{1 + 0.8000}\right)^3 = 421.24 \quad (17)$$

Using Equation (18) above and the data in Table 1 above, the value of the certainty equivalent at time zero is...

$$PVCE = 421.24 \times (1 + 0.0450)^{-3} = 369.13 \quad (18)$$

2. What is the value of the scalar alpha? Using Equation (5) above and the data in Table 1 above, the guess value of our scalar is...

$$\alpha = -\ln\left(1 - 0.80\right) / 950.00 = 0.001809 \quad (19)$$

Using Equations (16) and (17) above and the data in Table 1 above, the value of our scalar is...

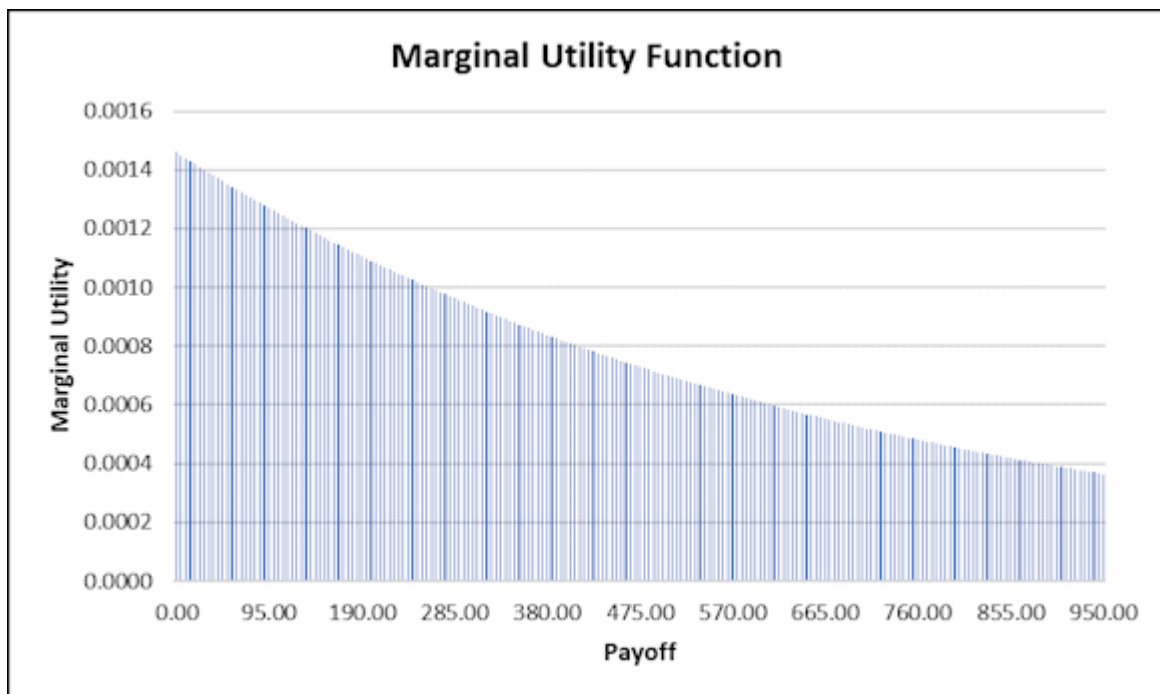
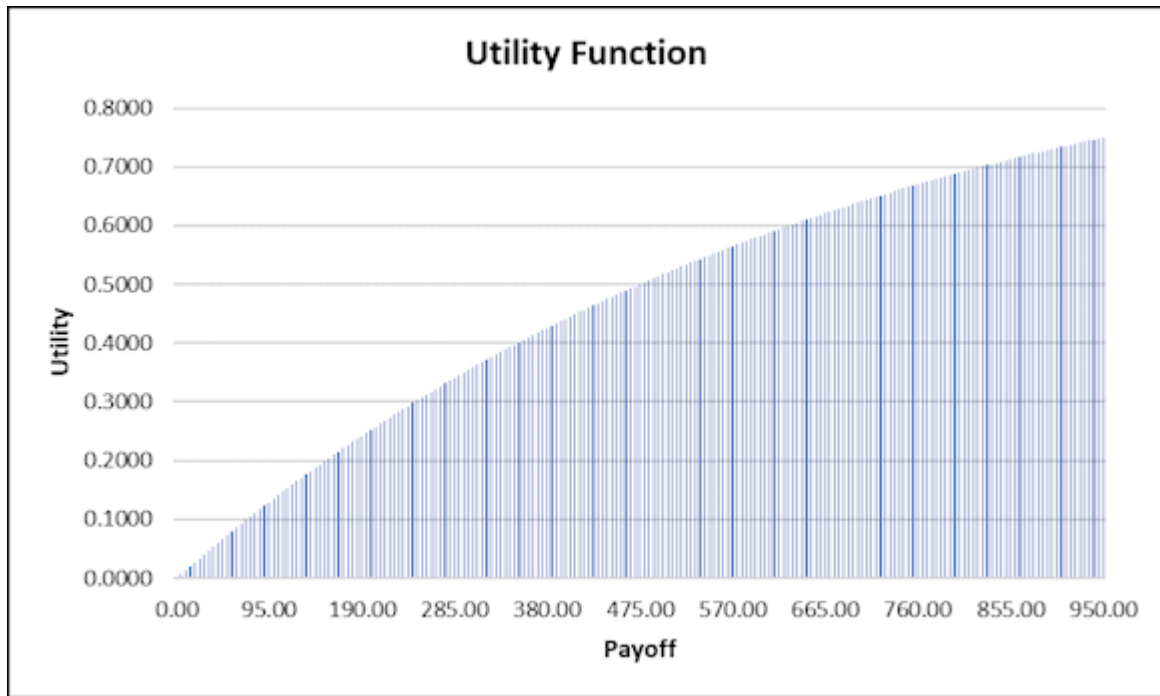
Iteration	Scalar	$f(\alpha)$	$f(\hat{\alpha})$	$f'(\hat{\alpha})$
0	0.001694	0.000000	0.004880	23.273
1	0.001484	0.000000	0.000134	21.830
2	0.001478	0.000000	0.000000	21.772
3	0.001478	0.000000	0.000000	21.772

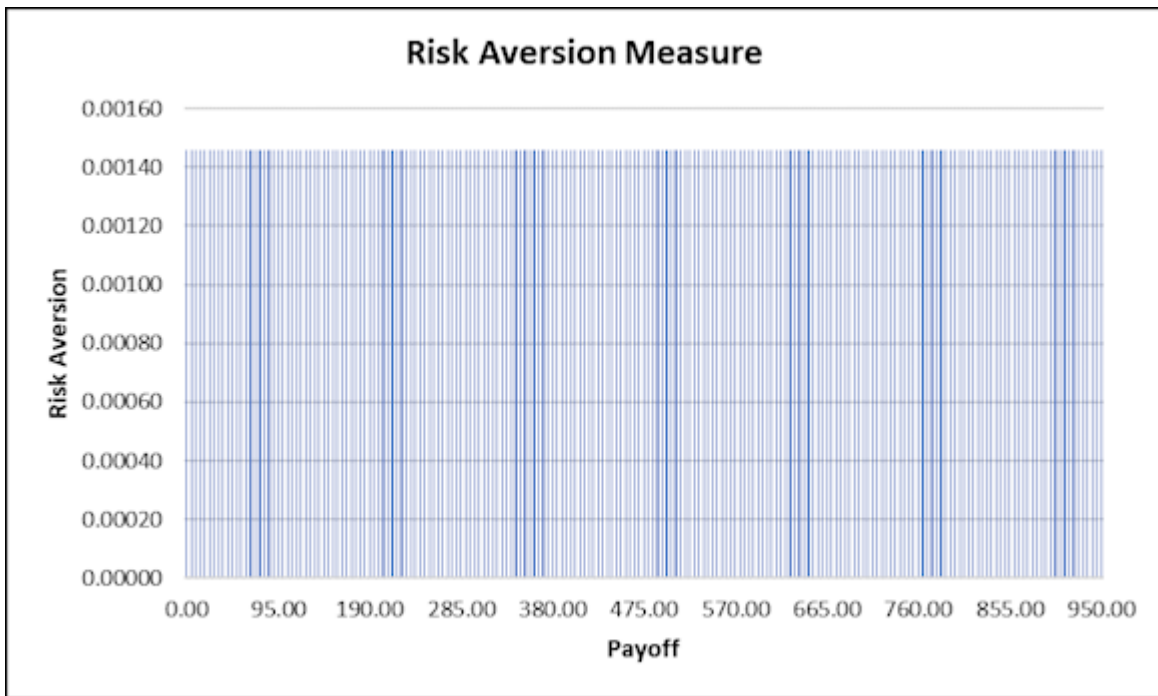
Using the results in the table above, the answer to the question is...

$$\text{if... } \alpha = \text{scalar value} = 0.001478 \text{ ...then... } \kappa = \text{investment rate of return} = 8.00\% \quad (20)$$

3. Graph utility, marginal utility, and risk aversion.

Using Equation (3) above and our estimate of the scalar parameter in Equation (20) above, our graphs are...





References

- [1] Gary Schurman, *Introduction To Utility Funtions*, October, 2023.
- [2] Gary Schurman, *The Newton-Raphson Method For Solving Non-Linear Equations*, October, 2009.